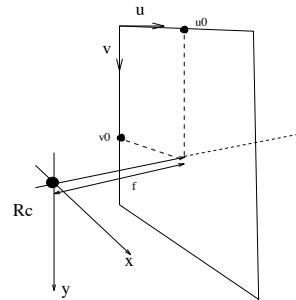


Geometry for Augmented Reality



T. Chateau



Content

1. Introduction
2. Camera models
3. Planar geometry
4. Introduction to 3D geometry

Homogeneous Coordinates

Homogeneous Coordinates

Homogeneous coordinates are a way of representing N-dimensional coordinates with N+1 numbers

Therefore, a point in Cartesian coordinates, (X, Y) becomes (x, y, w) in Homogeneous coordinates

$$\text{Homogeneous} \quad \text{Cartesian}$$
$$(x, y, w) \leftrightarrow \left(\frac{x}{w}, \frac{y}{w} \right)$$

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Homogeneous Coordinates

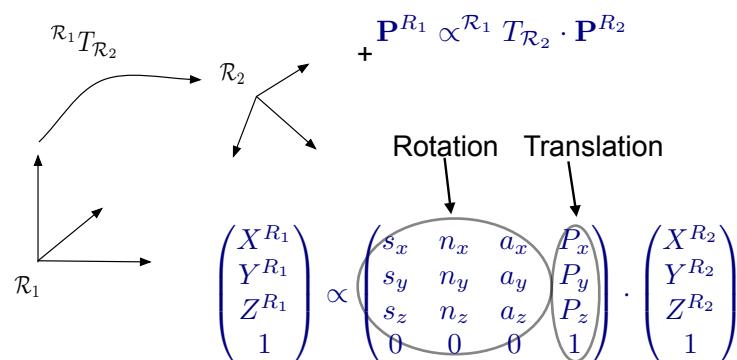
Solution: Homogeneous Coordinates

$$\begin{array}{ccc} \text{Homogeneous} & & \text{Cartesian} \\ (1, 2, 3) = (2, 4, 6) = (1a, 2a, 3a) & \rightarrow & \left(\frac{1}{3}, \frac{2}{3} \right) \end{array}$$

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Homogeneous Transformations

Homogeneous Transformations



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Homogeneous Transformations

Rigid Transformations

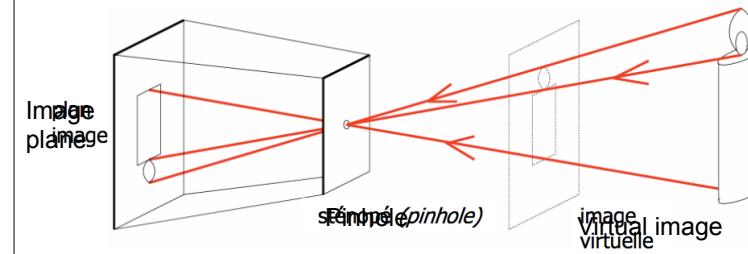
$$\begin{pmatrix} X^{R_1} \\ Y^{R_1} \\ Z^{R_1} \\ 1 \end{pmatrix} \propto \begin{pmatrix} s_x & n_x & a_x & P_x \\ s_y & n_y & a_y & P_y \\ s_z & n_z & a_z & P_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} X^{R_2} \\ Y^{R_2} \\ Z^{R_2} \\ 1 \end{pmatrix}$$
$$\mathbf{P}^{R_1} = \begin{pmatrix} R_1 R_{R_2} & R_1 \mathbf{T}_{R_2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X^{R_2} \\ Y^{R_2} \\ Z^{R_2} \\ 1 \end{pmatrix} = \begin{pmatrix} X^{R_1} \\ Y^{R_1} \\ Z^{R_1} \\ 1 \end{pmatrix}$$

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Camera Models

Pinhole model

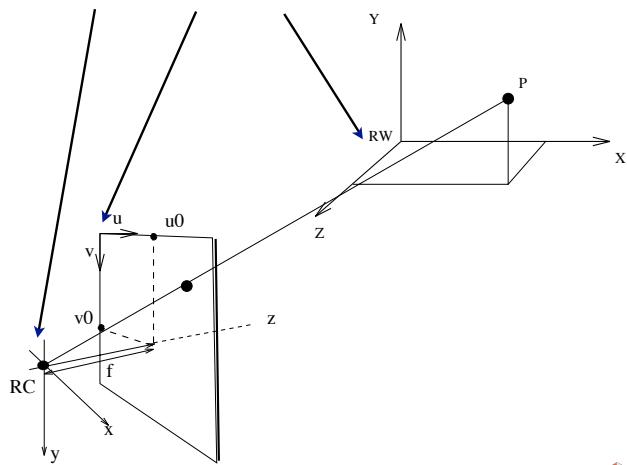


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Camera Models

Pinhole model: 3 reference frames

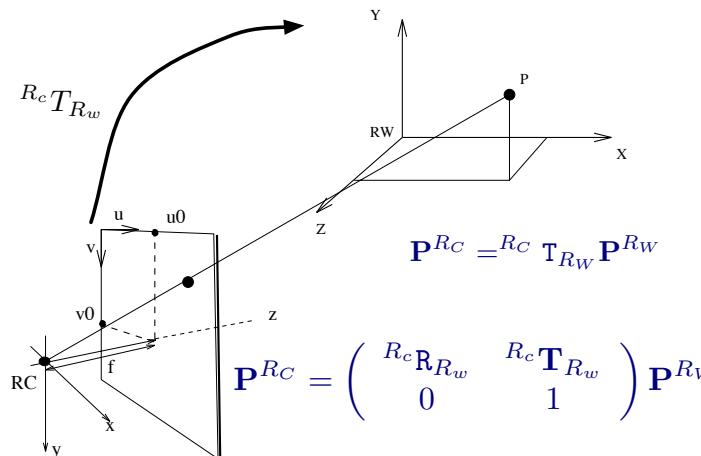


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Camera Models

Pinhole model: extrinsic parameter matrix

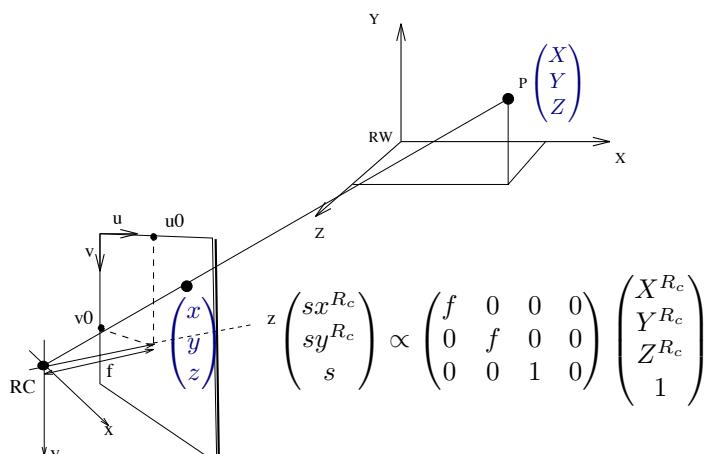


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Camera Models

Pinhole model: perspective projection

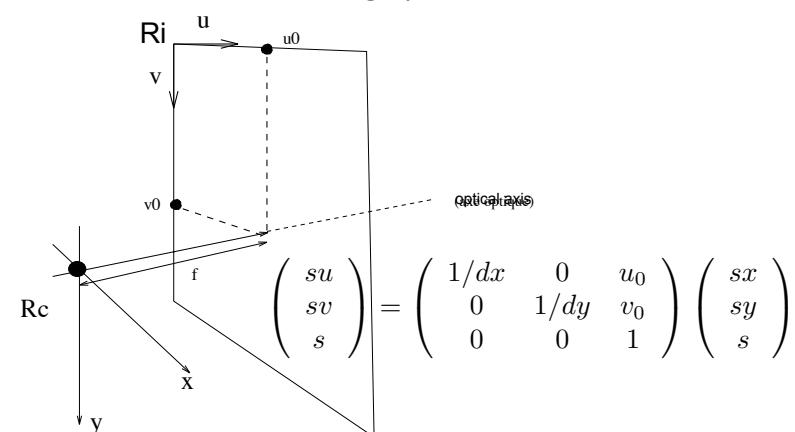


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Camera Models

Pinhole model: image plane transformation



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Camera Models

Pinhole model: image plane transformation

$$\begin{cases} u = x/dx + u_0 \\ v = y/dy + v_0 \end{cases}$$

$$\begin{pmatrix} su \\ sv \\ s \end{pmatrix} = \begin{pmatrix} 1/dx & 0 & u_0 \\ 0 & 1/dy & v_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} sx \\ sy \\ s \end{pmatrix}$$

- (u_0, v_0) are pixel based coordinates into the image of the intersection between the optical axis and the image plane
- (dx, dy) are the x and y dimension of one pixel of the physical sensor.

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Camera Models

Pinhole model: intrinsic parameter matrix

$$M_{int} = \begin{pmatrix} fx & 0 & u_0 \\ 0 & fy & v_0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{c} f_x \\ f_y \end{array} \right) = \left(\begin{array}{c} f \\ f \end{array} \right) \begin{array}{l} dx \\ dy \end{array} \quad \left. \right\} \rightarrow (f_x/f_y = dy/dx)$$

pixels mm

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Camera Models

Pinhole model: intrinsic parameter matrix

$$\begin{pmatrix} su \\ sv \\ s \end{pmatrix} = \begin{pmatrix} 1/dx & 0 & u_0 \\ 0 & 1/dy & v_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} X^{R_c} \\ Y^{R_c} \\ Z^{R_c} \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} su \\ sv \\ s \end{pmatrix} = \begin{pmatrix} f/dx & 0 & u_0 & 0 \\ 0 & f/dy & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} X^{R_c} \\ Y^{R_c} \\ Z^{R_c} \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} su \\ sv \\ s \end{pmatrix} = \begin{pmatrix} fx & 0 & u_0 & 0 \\ 0 & fy & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} X^{R_c} \\ Y^{R_c} \\ Z^{R_c} \\ 1 \end{pmatrix}$$

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Camera Models

Camera Models

Pinhole model: global projection matrix

$$\begin{pmatrix} su \\ sv \\ s \end{pmatrix} = M_{int} R_C T_{RW} \begin{pmatrix} X^{R_W} \\ Y^{R_W} \\ Z^{R_W} \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} su \\ sv \\ s \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{pmatrix} \begin{pmatrix} X^{R_W} \\ Y^{R_W} \\ Z^{R_W} \\ 1 \end{pmatrix}$$

P

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Camera Models

Pinhole model: global projection matrix

$$\begin{pmatrix} su \\ sv \\ s \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{pmatrix} \begin{pmatrix} X^{R_W} \\ Y^{R_W} \\ Z^{R_W} \\ 1 \end{pmatrix}$$

$$\begin{cases} u = \frac{m_{11}X^{R_w} + m_{12}Y^{R_w} + m_{13}Z^{R_w} + m_{14}}{m_{31}X^{R_w} + m_{32}Y^{R_w} + m_{33}Z^{R_w} + m_{34}} \\ v = \frac{m_{21}X^{R_w} + m_{22}Y^{R_w} + m_{23}Z^{R_w} + m_{24}}{m_{31}X^{R_w} + m_{32}Y^{R_w} + m_{33}Z^{R_w} + m_{34}} \end{cases}$$

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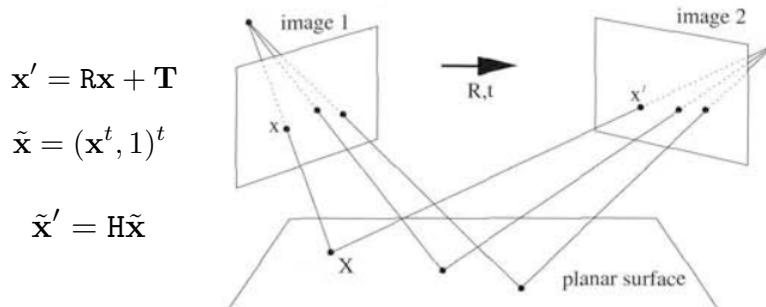
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Planar transformations

Homography

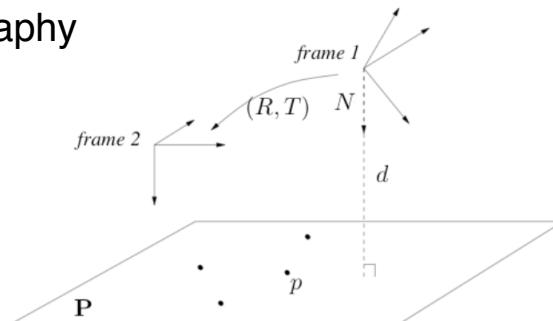


H : Homography (3x3 matrix)

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Planar transformations

Homography



$$H = R + \frac{1}{d}TN^T$$

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Planar transformations

Homography

$$\tilde{\mathbf{x}}' = \mathbf{H}\tilde{\mathbf{x}}$$

$$\tilde{\mathbf{x}}' \propto \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix} \tilde{\mathbf{x}}$$

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Planar transformations

Homography Estimation

$$\tilde{\mathbf{x}}' \propto \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix} \tilde{\mathbf{x}}$$

Build an estimator

$$\hat{\theta} = (h_{11}, h_{12}, h_{13}, h_{21}, h_{22}, h_{23}, h_{31}, h_{32}, h_{33})^t$$

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Planar transformations

Homography Estimation

$$\tilde{\mathbf{x}}' \propto \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix} \tilde{\mathbf{x}}$$

Build an estimator

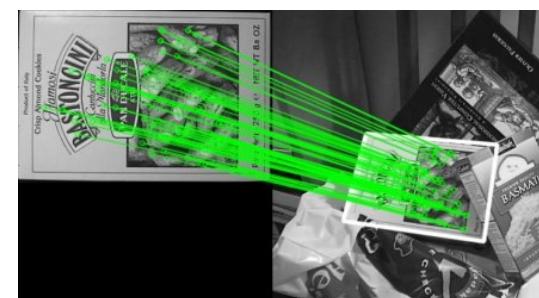
$$\hat{\theta} = (h_{11}, h_{12}, h_{13}, h_{21}, h_{22}, h_{23}, h_{31}, h_{32}, h_{33})^t$$

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Planar transformations

Homography estimation

We use a set of matches: $\{\mathbf{x}_n; \mathbf{x}'_n\}_{n=1,\dots,N}$



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Planar transformations

Exercice: Homography estimation

Given a set of matches: $\{\mathbf{x}_n; \mathbf{x}'_n\}_{n=1,\dots,N}$

$$\hat{\theta} = (h_{11}, h_{12}, h_{13}, h_{21}, h_{22}, h_{23}, h_{31}, h_{32}, h_{33})^t$$

Write the homography estimation problem as a linear system:

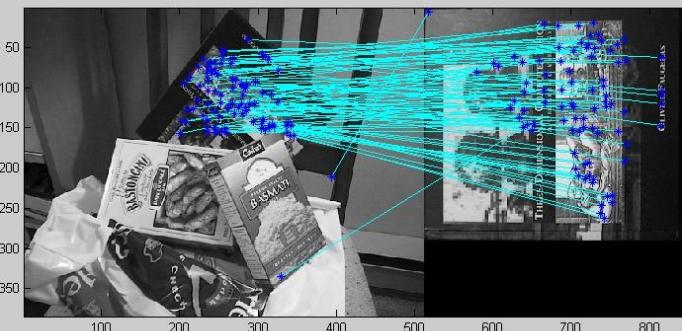
$$A.\hat{\theta} = B \text{ by setting } h_{33} = 1$$

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Planar transformations

Robust homography estimation:

OUTLIERS



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Planar transformations

Homography estimation: [Hartley]

$$Hx_i = \begin{pmatrix} h^{1T}x_i \\ h^{2T}x_i \\ h^{3T}x_i \end{pmatrix} \quad x'_i \times Hx_i = \begin{pmatrix} y'_i h^{3T}x_i - w'_i h^{2T}x_i \\ w'_i h^{1T}x_i - x'_i h^{3T}x_i \\ x'_i h^{2T}x_i - y'_i h^{1T}x_i \end{pmatrix}$$

$$\begin{bmatrix} 0^T & -w'_i x_i^T & y'_i x_i^T \\ w'_i x_i^T & 0^T & -x'_i x_i^T \\ -y'_i x_i^T & x'_i x_i^T & 0^T \end{bmatrix} \begin{pmatrix} h^1 \\ h^2 \\ h^3 \end{pmatrix} = 0.$$

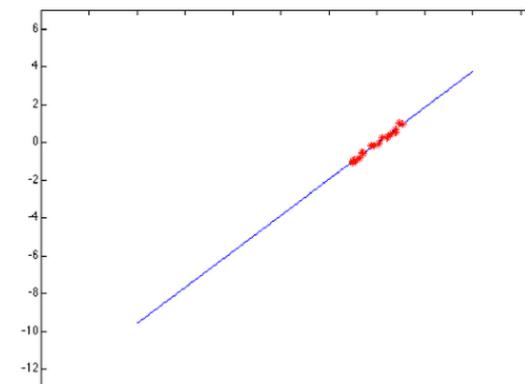
$$\begin{bmatrix} 0^T & -w'_i x_i^T & y'_i x_i^T \\ w'_i x_i^T & 0^T & -x'_i x_i^T \end{bmatrix} \begin{pmatrix} h^1 \\ h^2 \\ h^3 \end{pmatrix} = 0.$$

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Planar transformations

Robust homography estimation:

Least square approximation

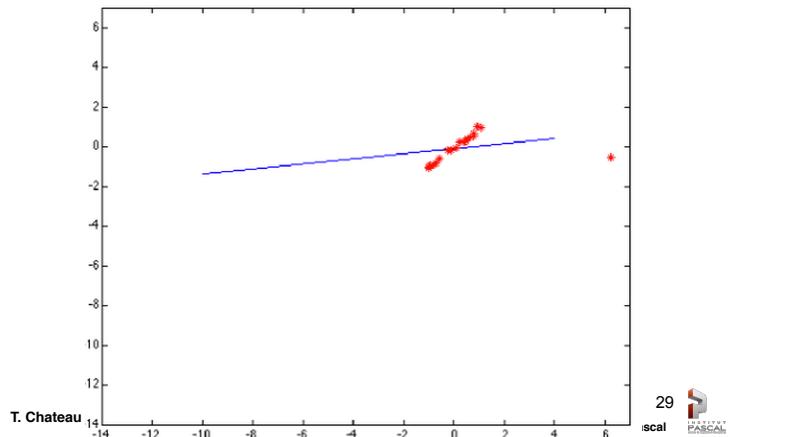


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Planar transformations

Robust homography estimation:

Least square approximation



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Planar transformations

Robust homography estimation:

RANSAC

RANdom SAmple Consensus

- Approach: we want to avoid the impact of outliers, so let's look for inliers only
- Intuition: if an outlier is chosen to compute the current fit, then the resulting line won't have much support from the rest of the points

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Planar transformations

Robust homography estimation:

RANSAC

RANdom SAmple Consensus

- Approach: we want to avoid the impact of outliers, so let's look for inliers only
- Intuition: if an outlier is chosen to compute the current fit, then the resulting line won't have much support from the rest of the points

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Planar transformations

Robust homography estimation:

RANSAC

RANdom SAmple Consensus

- Approach: we want to avoid the impact of outliers, so let's look for inliers only
- Intuition: if an outlier is chosen to compute the current fit, then the resulting line won't have much support from the rest of the points

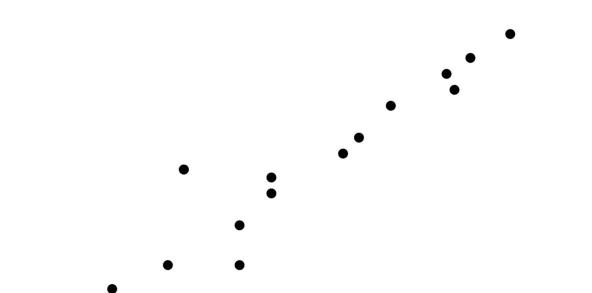
Planar transformations

Robust homography estimation:

RANSAC

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Planar transformations

Robust homography estimation:

RANSAC

Sample two points

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Planar transformations

Robust homography estimation:

RANSAC

Count the number of inliers

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Robust homography estimation:

RANSAC

Fit a line through them

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Robust homography estimation:

RANSAC

Repeat, until we get a good result

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Planar transformations

Robust homography estimation:

RANSAC: How many samples?

- How many samples are needed?
 - Suppose w is fraction of inliers (points from line)
 - n points needed to define hypothesis (2 for lines)
 - k samples chosen
- Probability that a single sample of n points is correct: w^n
- Probability that all samples fail is: $(1 - w^n)^k$
- Choose k high enough to keep this below desired failure rate.

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Planar transformations

Robust homography estimation:

RANSAC

Echantillons	% outliers						
	5%	10%	20%	25%	30%	40%	50%
2	2	3	5	6	7	11	17
3	3	4	7	9	11	19	35
4	3	5	9	13	17	34	72
5	4	6	12	17	26	57	146
6	4	7	16	24	37	97	293
7	4	8	20	33	54	163	588
8	5	9	26	44	78	272	1177

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Planar transformations

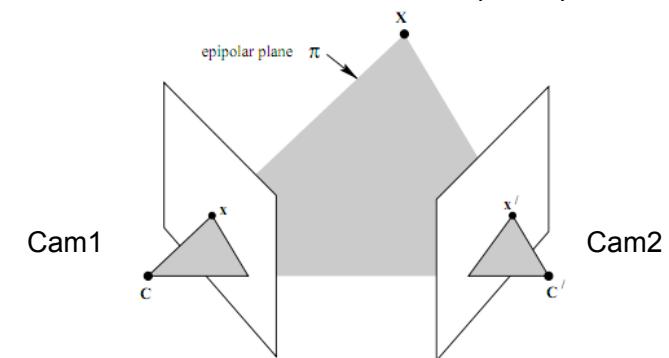
Robust homography estimation: exercice

Write a RANSAC Algorithm to estimate an homography. We assume that the outlier rate is up to 50%.

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3D Vision

Multi-view 3D reconstruction (intro)

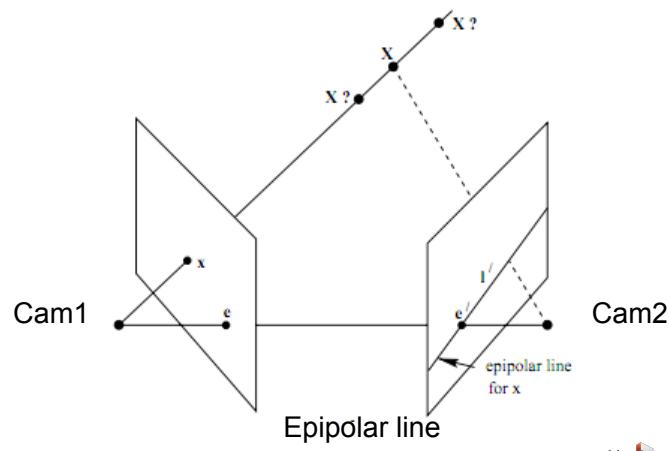


Epipolar plane

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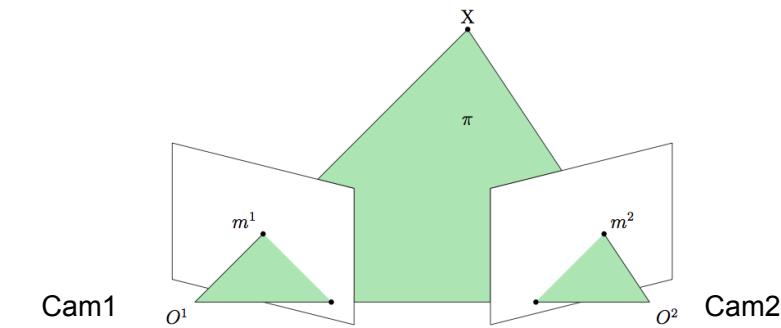
3D Vision

Multi-view 3D reconstruction (intro)



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3D Vision



$$\overrightarrow{O^1m^1} \cdot [\overrightarrow{O^2m^2} \times \overrightarrow{O^1O^2}] = 0$$

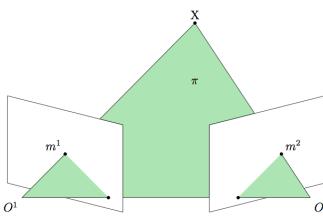
\Leftrightarrow

$$m^1 \cdot [T \times (R \cdot m^2)] = 0$$

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3D Vision

Essential matrix



$$\overrightarrow{O^1m^1} \cdot [\overrightarrow{O^2m^2} \times \overrightarrow{O^1O^2}] = 0$$

$$\Leftrightarrow m^1 \cdot [T \times (R \cdot m^2)] = 0$$

\Leftrightarrow

$$\Leftrightarrow$$

$$T_x = \begin{pmatrix} 0 & t_3 & -t_2 \\ -t_3 & 0 & t_1 \\ t_2 & -t_1 & 0 \end{pmatrix}$$

3D Vision

Essential matrix

Estimation of essential matrix can be done using:

- 5 matches: 5-points algorithm
 - 8 matches: 8-points algorithm

See Nister's work for further information

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3D Vision

Fondamental (Essential) Matrix

The fondamental matrix is the key relation in structure from motion algorithms.

Further reading in:

```
@Book{Hartley2000,
  author = "Hartley, R.-I. and Zisserman, A.",
  title = "Multiple View Geometry in Computer
Vision",
  year = "2000",
  publisher = "Cambridge University Press, ISBN:
0521623049"
}
```

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Exercices

Stereo

Two calibrated cameras see the same 3D rigid scene.

- 1) Write the linear system which gives the estimation of a 3D point from the projection of the point in the two calibrated cameras (in a least square way).

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